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An approximate solution scheme for the algebraic random eigenvalue problem

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Abstract

A reduced basis formulation is presented for the efficient solution of large-scale algebraic random eigenvalue problems. This formulation aims to improve the accuracy of the first order perturbation method, and also allow the efficient computation of higher order statistical moments of the eigenparameters. In the present method, the two terms of the first order perturbation approximation for the eigenvector are used as basis vectors for Ritz analysis of the governing random eigenvalue problem. This leads to a sequence of reduced order random eigenvalue problems to be solved for each eigenmode of interest. Since, only two basis vectors are used to represent each eigenvector, explicit expressions for the random eigenvalues and eigenvectors can readily be derived. This enables the statistics of the random eigenparameters and the forced response to be efficiently computed. Numerical studies are presented for free and forced vibration analysis of a linear stochastic structural system. It is demonstrated that the reduced basis method gives better results as compared to the first order perturbation method.

1. Introduction

Linear stochastic differential eigenvalue problems (SDEPs) are frequently encountered in the entire spectrum of science and engineering; for example, structural dynamics, stability analysis, quantum chemistry, and electrical networks. It is known that spatial discretization techniques can be used in conjunction with random field discretization schemes (or random variable models of uncertainty) to represent a linear SDEP in a finite-dimensional setting as an algebraic random eigenvalue problem. These representation schemes have been widely used in the stochastic finite

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element method, wherein randomness is treated as an additional dimension of the problem; see, for example, Ref. [1].

In many problems of practical interest, the size of the discretized equations poses a formidable obstacle to the application of Monte Carlo simulation (MCS) schemes for accurately estimating the statistics of the eigenvalues and eigenvectors. Hence, in order to compute the eigenparameter statistics in a computationally efficient fashion, the development of approximate solution schemes has been pursued with particular vigor in the computational stochastic mechanics literature. The application areas of solution techniques for random eigenvalue problems include stochastic structural dynamics [2,3], robustness analysis of structural and control systems [4], structural model updating and damage identification [5], and parameter-based statistical energy analysis [6].

The approaches in the literature for tackling algebraic random eigenvalue problems can be broadly categorized into non-parametric and parametric techniques. The term non-parametric is used here to refer to techniques which directly model the terms of the coefficient matrices as random variables; see, for example, Ref. [7]. In the domain of structural dynamics, non-parametric approaches which directly postulate a probabilistic model for the natural frequencies and mode shapes have also been considered; see Ref. [8] and the references therein. Based on some assumptions on the statistics of the natural frequencies, Langley showed that a compact expression can be derived for the variance of the frequency response. This class of approaches do not directly take into account the primary sources of the uncertainties, such as the random system parameters. As a consequence, one has to prove by example that the assumed statistical model for the eigenparameters leads to reasonable approximations for the forced response statistics.

The present paper is concerned with parametric approaches for solving the algebraic random eigenvalue problem, where the coefficient matrices can be represented as a linear combination of the random system parameters. The underlying assumption made here is that the system uncertainty has an underlying structure. This representation of uncertainty is not limiting. As discussed earlier, the advances made in the area of stochastic finite element analysis make this representation of uncertainty possible for many systems of practical interest. Note that parametric approaches may also be generalized to problems where the terms of the coefficient matrices can be represented as polynomials in the random system parameters. Further, parametric techniques can be employed to validate the assumptions used in non-parametric approaches to stochastic structural dynamic analysis.

In 1969, Collins and Thompson [9] presented a first order perturbation method for dynamic analysis of structures with parameter uncertainties. A detailed overview of the perturbation method for algebraic random eigenvalue problems can be found in the monograph of Kleiber and Hien [10]. A recent review of the state of the art in stochastic structural dynamics [2] suggests that the first order perturbation method appears to be the most widely used approach for approximating the statistics of the eigenparameters. The popular use of this method can be primarily attributed to ease of implementation and computational efficiency. However, the perturbation method only gives reasonable quality results for the statistical moments when the coefficients of variation of the random system parameters are small. Further, since the higher order perturbation terms are computationally intensive to compute, it is often difficult to improve the accuracy of first order approximations in practice.

Lee and Singh [11] presented an approximate method based on direct matrix products for approximating the first two statistical moments of the eigenvalues and eigenvectors. This method

can be interpreted as a zero order Rayleigh quotient approximation procedure. As a result, for the case when only the stiffness matrix is random, this method reduces to the first order perturbation method for the eigenvalues. It was demonstrated for some simple example problems that improvements over the first order perturbation method can be achieved. Approaches which focus only on approximating the statistics of the eigenvalues have also been proposed in the literature; see, for example, Ref. [12].

More recently, a response surface method combining the polynomial chaos expansion and the MCS was proposed by Red-Horse and Ghanem [13]. In this method, the eigenvalues and eigenvectors are represented by a polynomial chaos expansion scheme. The coefficients in the expansion are then evaluated as generalized Fourier coefficients via the MCS procedure. This representation allows the computation of additional statistics of the eigensolution in an efficient fashion. The main drawback of this approach is the requirement of MCS, which is computationally expensive for large-scale eigenvalue problems.

In order to reduce the computational cost of MCS for large-scale problems, the application of model reduction schemes has been investigated; see, for example, Ref. [14]. However, the computational cost savings using this approach may not be significant for problems where the statistics of a large number of eigenmodes are to be computed. In contrast, the present formulation involves the construction of a sequence of reduced order problems for each eigenmode of interest. This is expected to lead to better efficiency when the statistics of a large number of eigenmodes are to be computed.

Stochastic reduced basis methods (SRBMs) for numerical solution of systems governed by stochastic partial differential equations have been proposed by the authors; see, for example, Refs. [15–17]. This class of algorithms is intended for problems where discretization of the governing equations in space together with the random dimension ultimately leads to a linear random algebraic system of equations. The formulation developed in the present paper is similar in spirit to SRBMs. However, the choice of stochastic basis vectors and the details of the formulation are different. The choice of basis vectors used in the present formulation is motivated by a method proposed earlier in Ref. [18,19] for structural dynamic reanalysis.

So, to summarize, the focus of the present research is on developing a computationally efficient numerical scheme for solving large-scale algebraic random eigenvalue problems. Procedures for discretizing linear SDEPs in space and the random dimension to arrive at an algebraic random eigenvalue problem are outlined. The two terms of the first order perturbation approximation for the eigenvector are chosen as basis vectors in conjunction with undetermined random functions for representing the random eigenvector of the discretized SDEP. The undetermined random functions in the reduced basis representation are computed via Ritz analysis of the random eigenvalue problem. This leads to a sequence of (2×2) reduced order random eigenvalue problems for each eigenmode of interest. Explicit expressions for the random eigenvalues and eigenvectors are derived in terms of the random variables arising from discretization of the underlying random fields or the random system parameters. This enables a complete statistical description of the eigenvalues and eigenvectors in a computationally efficient fashion.

Numerical studies are presented for free and forced vibration analysis of a network of Euler–Bernoulli beams with random Young's modulus. It is shown that the present method gives better results compared to the first order perturbation method.

2. Problem statement

Consider a linear stochastic differential eigenvalue problem of the form

$$\mathscr{K}(\alpha(\mathbf{x},\boldsymbol{\theta}))[u(\mathbf{x},\boldsymbol{\theta})] = \lambda(\boldsymbol{\theta})\mathscr{M}(\beta(\mathbf{x},\boldsymbol{\theta}))[u(\mathbf{x},\boldsymbol{\theta})],$$

where $\mathscr{K}(\alpha(\mathbf{x},\boldsymbol{\theta}))$ and $\mathscr{M}(\alpha(\mathbf{x},\boldsymbol{\theta}))$ are stochastic differential operators defined on the domain $\mathscr{D}$; $\mathbf{x}\in\mathscr{D}$ denotes a point on the domain; $\boldsymbol{\theta}\in\Omega$ belongs to the Hilbert space of second order random variables; $\alpha(\mathbf{x},\boldsymbol{\theta})$ and $\beta(\mathbf{x},\boldsymbol{\theta})$ are second order random fields describing the coefficients of the stochastic differential operators; $\lambda(\boldsymbol{\theta})$ and $u(\mathbf{x},\boldsymbol{\theta})$ are the random eigenvalues and eigenfunctions, respectively.

To illustrate the notation used here, consider the SDEP governing the flexural free-vibrations of a stochastic Euler–Bernoulli beam. Here, the stochastic differential operators $\mathscr{K}$ and $\mathscr{M}$ can be defined as

$$\mathscr{K} = \frac{\partial^2}{\partial x^2} EI(\mathbf{x},\boldsymbol{\theta}) \frac{\partial^2}{\partial x^2} \quad \text{and} \quad \mathscr{M} = m(\mathbf{x},\boldsymbol{\theta}), \tag{1}$$

where $EI(\mathbf{x},\boldsymbol{\theta})$ and $m(\mathbf{x},\boldsymbol{\theta})$ are random fields describing the flexural rigidity and mass density, respectively.

The random fields describing the coefficients of the differential operators can be discretized using techniques available in the literature, such as Karhunen–Loéve (KL) expansion, polynomial chaos expansion, and optimal linear estimation; see, for example, Ref. [20]. Random field discretization involves the representation of the field in terms of a finite number of random variables, which are amenable to a numerical treatment. Consider the case when the random field $\alpha(\mathbf{x},\boldsymbol{\theta})$ is discretized using the mean-square convergent KL expansion scheme [1] as shown below:

$$\alpha(\mathbf{x},\boldsymbol{\theta}) = \langle \alpha(\mathbf{x},\boldsymbol{\theta}) \rangle + \sum_{i=0}^{\infty} \eta_i^{\alpha} \sqrt{\mu_i^{\alpha}}\, \alpha_i(\mathbf{x}), \tag{2}$$

where μ_i^{α} and $\alpha_i(\mathbf{x})$ are the characteristic functions (eigenvalues and eigenvectors, respectively) of the following deterministic integral eigenvalue problem:

$$\mu_i^{\alpha}\alpha_i(\mathbf{x}) = \int_{\Omega} R_{\alpha\alpha}(\mathbf{x},\mathbf{x}_1)\alpha_i(\mathbf{x}_1)\,\mathrm{d}\mathbf{x}_1, \tag{3}$$

where $R_{\alpha\alpha}$ is the correlation function of the random field. The vector of zero-mean random variables $\{\eta_i^{\alpha}\}$ are uncorrelated, i.e., $\langle \eta_i^{\alpha}\eta_j^{\alpha} \rangle = \mu_i^{\alpha}\delta_{ij}$, where δ_{ij} denotes the Kronecker delta function. Analytical solutions for the characteristic functions of Eq. (3) can be readily computed for a class of correlation functions defined on simple domains. Further details including approximate schemes for numerical solution of Eq. (3) for complex domains can be found in the literature; see, for example, Ref. [1,21]. The KL expansion of the random field $\beta(\mathbf{x},\boldsymbol{\theta})$ can be carried out similarly.

In practice, depending on the correlation length of the random fields, a small number of terms from the KL expansion can be used to represent the underlying random fields without significant loss of accuracy. Using the KL expansions of the random fields, the stochastic differential operators in Eq (1) can be written as the sum of a deterministic and stochastic operator as

$$(\mathscr{K}^o + \mathscr{K}_\theta)[u(\mathbf{x},\boldsymbol{\theta})] = \lambda(\boldsymbol{\theta})(\mathscr{M}^o + \mathscr{M}_\theta)[u(\mathbf{x},\boldsymbol{\theta})], \tag{4}$$

where $\mathscr{K}^o$ and $\mathscr{M}^o$ are deterministic differential operators, and $\mathscr{K}_\theta$ and $\mathscr{M}_\theta$ are stochastic differential operators.

For the simplicity of presentation, consider the case when the random coefficients of the stochastic differential operators appear as multiplicative terms.[1] Further, let the random variables arising from discretization of the random fields be denoted by the vector $\boldsymbol{\theta} = \{\theta_i\}$, $i = 1, 2, \ldots, p$. Hence, without any loss of generality, Eq. (4) can be rewritten as

$$\left(\mathscr{K}^o + \sum_{i=1}^{p} \theta_i \mathscr{K}_i\right)[u(\mathbf{x}, \boldsymbol{\theta})] = \lambda(\boldsymbol{\theta})\left(\mathscr{M}^o + \sum_{i=1}^{p} \theta_i \mathscr{M}_i\right)[u(\mathbf{x}, \boldsymbol{\theta})], \tag{5}$$

where $\mathscr{K}_i$ and $\mathscr{M}_i$ are deterministic differential operators.

A spatial discretization technique such as the finite element method (FEM) can be used to represent Eq. (5) as an algebraic random eigenvalue problem of the form

$$\left[\mathbf{A}_o + \sum_{i=1}^{p} \theta_i \mathbf{A}_i\right]\mathbf{x}(\boldsymbol{\theta}) = \lambda(\boldsymbol{\theta})\left[\mathbf{B}_o + \sum_{i=1}^{p} \theta_i \mathbf{B}_i\right]\mathbf{x}(\boldsymbol{\theta}), \tag{6}$$

where $\mathbf{A}_o$, $\mathbf{B}_o$, $\mathbf{A}_i$, $\mathbf{B}_i \in \mathbb{R}^{n\times n}$ are deterministic matrices while $\lambda(\boldsymbol{\theta})$ and $\mathbf{x}(\boldsymbol{\theta}) \in \mathbb{R}^n$ denote the random eigenvalue and eigenvector, respectively.

For problems where the coefficients α and β appear non-linearly in the differential operators, either the Taylor series expansion or the polynomial chaos decomposition scheme [1] can be employed to arrive at a form similar to Eq. (6). A form similar to Eq. (6) can be also be readily arrived at for cases where the stochastic system properties are modelled as random variables. Here, the matrices $\mathbf{A}_i$ and $\mathbf{B}_i$ denote the sensitivities of the system matrices with respect to the random system parameters.

3. First order perturbation method

Consider the case when the differential operators in Eq. (1) are self-adjoint, and the matrices in Eq. (6) are symmetric positive definite. Further, let λ_o and $\mathbf{x}_o$ denote the eigenvalue and eigenvector, respectively, of the following deterministic eigenvalue problem

$$\mathbf{A}_o\mathbf{x}_o = \lambda_o\mathbf{B}_o\mathbf{x}_o. \tag{7}$$

The eigenvector of Eq. (7) can be normalized with respect to the matrix $\mathbf{B}_o$, i.e., $\mathbf{x}_o^{\mathrm{T}}\mathbf{B}_o\mathbf{x}_o = 1$. Note that for simplicity of presentation, the eigenmode numbers are not explicitly shown in the equations that follow. First order approximations for the random eigenvalue and eigenvector based on the deterministic eigenparameters of Eq. (7) can be written as

$$\tilde{\lambda}(\boldsymbol{\theta}) = \lambda_o + \sum_{j=1}^{p} \theta_j \frac{\partial\lambda}{\partial\theta_j}, \tag{8}$$

$$\tilde{\mathbf{x}}(\boldsymbol{\theta}) = \mathbf{x}_o + \sum_{j=1}^{p} \theta_j \frac{\partial\mathbf{x}}{\partial\theta_j}, \tag{9}$$

[1] For example, when Young's modulus of a structural member is modelled probabilistically.

where $\partial\lambda/\partial\theta_j$ and $\partial\mathbf{x}/\partial\theta_j$ are the sensitivities of the eigenvalues and eigenvectors with respect to the random variables, respectively. The eigenvalue and eigenvector derivatives can be computed using the equations

$$\frac{\partial\lambda}{\partial\theta_j} = \mathbf{x}_o^{\mathrm{T}}(\mathbf{A}_j - \lambda_o\mathbf{B}_j)\mathbf{x}_o \tag{10}$$

and

$$(\mathbf{A}_o - \lambda_o\mathbf{B}_o)\frac{\partial\mathbf{x}}{\partial\theta_j} = \left(\lambda_o\mathbf{B}_j + \frac{\partial\lambda}{\partial\theta_j}\mathbf{B}_o - \mathbf{A}_j\right)\mathbf{x}_o. \tag{11}$$

There exists a wealth of methods in the literature for solving Eq. (11); see, for example, Refs. [22,23]. In the present study, Akgün's first order method [24] is employed to approximately solve Eq. (11) and compute the eigenvector derivatives. Note that this formulation considers the eigenvalues of Eq. (7) to be distinct.

4. Stochastic reduced basis approximations

The fundamental idea of the present formulation is to use the two terms of the first order perturbation approximation (see Eq. (9)) as basis vectors for representing the eigenvector of the random eigenvalue problem. Note that this idea has been applied earlier with a degree of success in structural dynamic reanalysis [18,19]. The assumption made here is that the random eigenvector $\mathbf{x}(\boldsymbol{\theta})$ can be well approximated in the subspace spanned by $\mathbf{x}_o$ and $\sum_{i=1}^{p}\theta_i\partial\mathbf{x}/\partial\theta_i$, i.e., an approximation for $\mathbf{x}(\boldsymbol{\theta})$ can be written as

$$\hat{\mathbf{x}}(\boldsymbol{\theta}) = \zeta_1(\boldsymbol{\theta})\mathbf{x}_o + \zeta_2(\boldsymbol{\theta})\sum_{i=1}^{p}\theta_i\frac{\partial\mathbf{x}}{\partial\theta_j}, \tag{12}$$

where $\zeta_1(\boldsymbol{\theta})$ and $\zeta_2(\boldsymbol{\theta})$ are undetermined random functions.

To compute the undetermined functions $\zeta_1(\boldsymbol{\theta})$ and $\zeta_2(\boldsymbol{\theta})$, Eq. (12) is used for Ritz analysis of Eq. (6), which leads to a (2×2) random eigenvalue problem for each eigenmode of interest. The

Table 1
Elements of problem specific tensors

$a_i = \mathbf{x}_o^{\mathrm{T}}\mathbf{A}_i\mathbf{x}_o$	$b_i = \mathbf{x}_o^{\mathrm{T}}\mathbf{B}_i\mathbf{x}^o$
$c_i = \mathbf{x}_o^{\mathrm{T}}\mathbf{A}_o\dfrac{\partial\mathbf{x}}{\partial\theta_i}$	$d_i = \mathbf{x}_o^{\mathrm{T}}\mathbf{B}_o\dfrac{\partial\mathbf{x}}{\partial\theta_i}$
$E_{ij} = \mathbf{x}_o^{\mathrm{T}}\mathbf{A}_i\dfrac{\partial\mathbf{x}}{\partial\theta_j}$	$F_{ij} = \mathbf{x}_o^{\mathrm{T}}\mathbf{B}_i\dfrac{\partial\mathbf{x}}{\partial\theta_j}$
$G_{ij} = \dfrac{\partial\mathbf{x}^{\mathrm{T}}}{\partial\theta_i}\mathbf{A}_o\dfrac{\partial\mathbf{x}}{\partial\theta_j}$	$H_{ij} = \dfrac{\partial\mathbf{x}^{\mathrm{T}}}{\partial\theta_i}\mathbf{B}_o\dfrac{\partial\mathbf{x}}{\partial\theta_j}$
$\mathscr{Q}_{ijk} = \dfrac{\partial\mathbf{x}^{\mathrm{T}}}{\partial\theta_j}\mathbf{A}_i\dfrac{\partial\mathbf{x}}{\partial\theta_k}$	$\mathscr{R}_{ijk} = \dfrac{\partial\mathbf{x}^{\mathrm{T}}}{\partial\theta_j}\mathbf{B}_i\dfrac{\partial\mathbf{x}}{\partial\theta_k}$

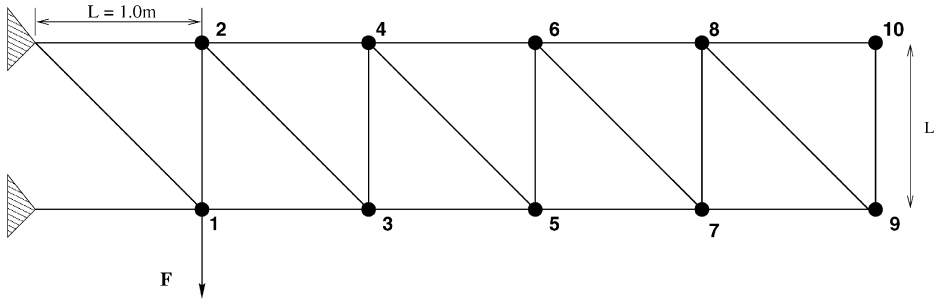

Fig. 1. Network of 20 Euler–Bernoulli beams.

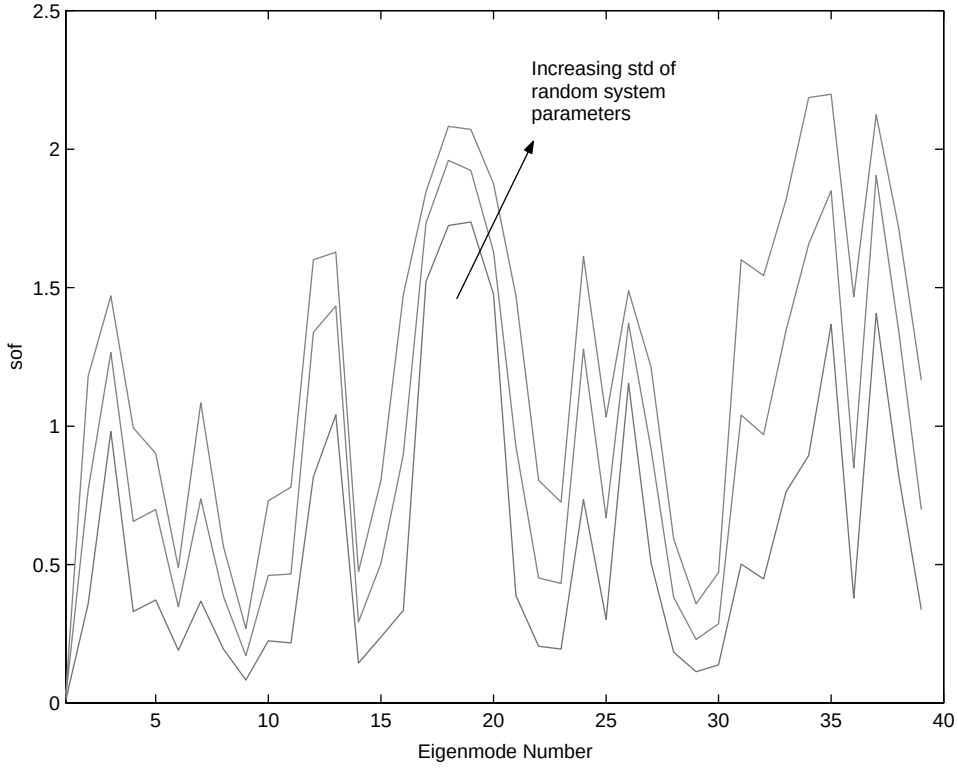

Fig. 2. Trends of the sof when the standard deviation of the random system parameters is increased from 0.05 to 0.15.

reduced order random eigenvalue problem can be written as

$$\mathbf{A}_R(\boldsymbol{\theta})\mathbf{Z}(\boldsymbol{\theta}) = \lambda(\boldsymbol{\theta})\mathbf{B}_R(\boldsymbol{\theta})\mathbf{Z}(\boldsymbol{\theta}), \tag{13}$$

where

$$\mathbf{A}_R(\boldsymbol{\theta}) = \boldsymbol{\Psi}^{\mathrm{T}}(\boldsymbol{\theta})\left[\mathbf{A}_o + \sum_{i=1}^{p} \theta_i \mathbf{A}_i\right]\boldsymbol{\Psi}(\boldsymbol{\theta}) \in \mathbb{R}^{2\times 2}, \tag{14}$$

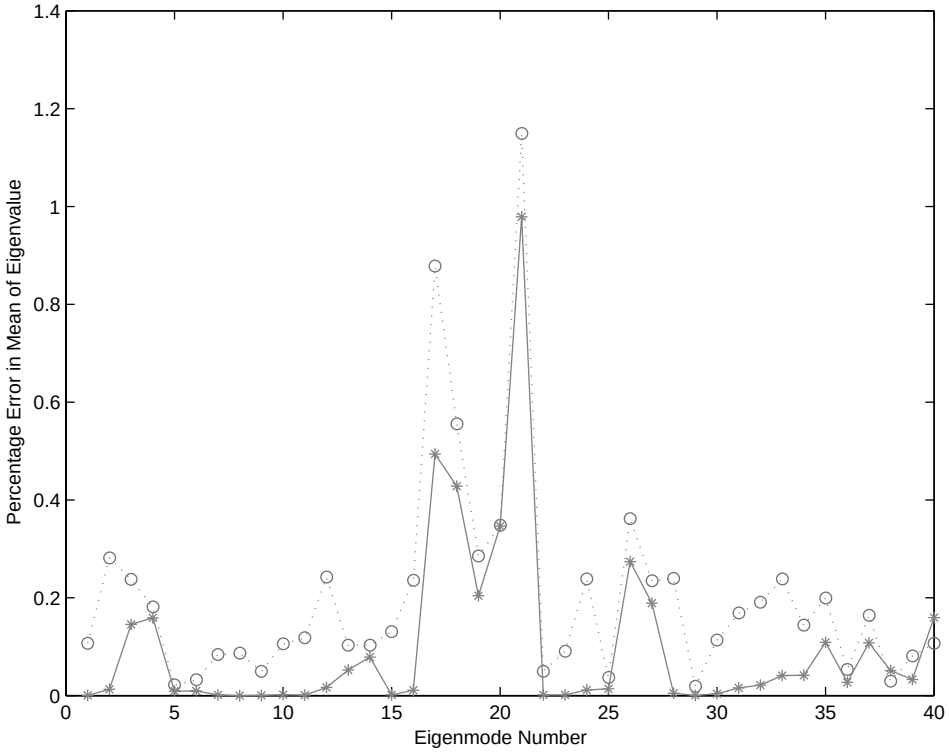

Fig. 3. Comparison of errors in mean of natural frequencies for case 1, $\sigma_\theta = 0.05$. The circles and stars represents precentage errors computed using PM1 and RBA, respectively. -o-, PM1; -∗-, RBA.

$$\mathbf{B}_R(\boldsymbol{\theta}) = \boldsymbol{\Psi}^{\mathrm{T}}(\boldsymbol{\theta})\left[\mathbf{B}_o + \sum_{i=1}^{p} \theta_i \mathbf{B}_i\right]\boldsymbol{\Psi}(\boldsymbol{\theta}) \in \mathbb{R}^{2\times 2}, \tag{15}$$

$$\boldsymbol{\Psi}(\boldsymbol{\theta}) = \left[\mathbf{x}^o, \sum_{i=1}^{p} \theta_i \frac{\partial \mathbf{x}}{\partial \theta_i}\right] \in \mathbb{R}^{n\times 2}, \tag{16}$$

and $Z = \{\zeta_1(\boldsymbol{\theta}), \zeta_2(\boldsymbol{\theta})\}^{\mathrm{T}} \in \mathbb{R}^2$. After some algebra, the elements of the reduced order random matrices $\mathbf{A}_R(\boldsymbol{\theta})$ and $\mathbf{B}_R(\boldsymbol{\theta})$ can be written using tensor notation as

$$\mathbf{A}_R(\boldsymbol{\theta}) = \begin{bmatrix} (\lambda_o + \theta_i a_i) & (\theta_i c_i + \theta_i\theta_j E_{ij}) \\ \text{sym} & (\theta_i\theta_j G_{ij} + \theta_i\theta_j\theta_k \mathscr{Q}_{ijk}) \end{bmatrix} \tag{17}$$

and

$$\mathbf{B}_R(\boldsymbol{\theta}) = \begin{bmatrix} (1 + \theta_i b_i) & (\theta_i d_i + \theta_i\theta_j F_{ij}) \\ \text{sym} & (\theta_i\theta_j H_{ij} + \theta_i\theta_j\theta_k \mathscr{R}_{ijk}) \end{bmatrix}. \tag{18}$$

Expressions for the deterministic terms a, b, c, d, E, F, G, H, $\mathscr{Q}$, and $\mathscr{R}$ are given in Table 1. In the notation used here, repeated indices imply summation with respect to that index over the

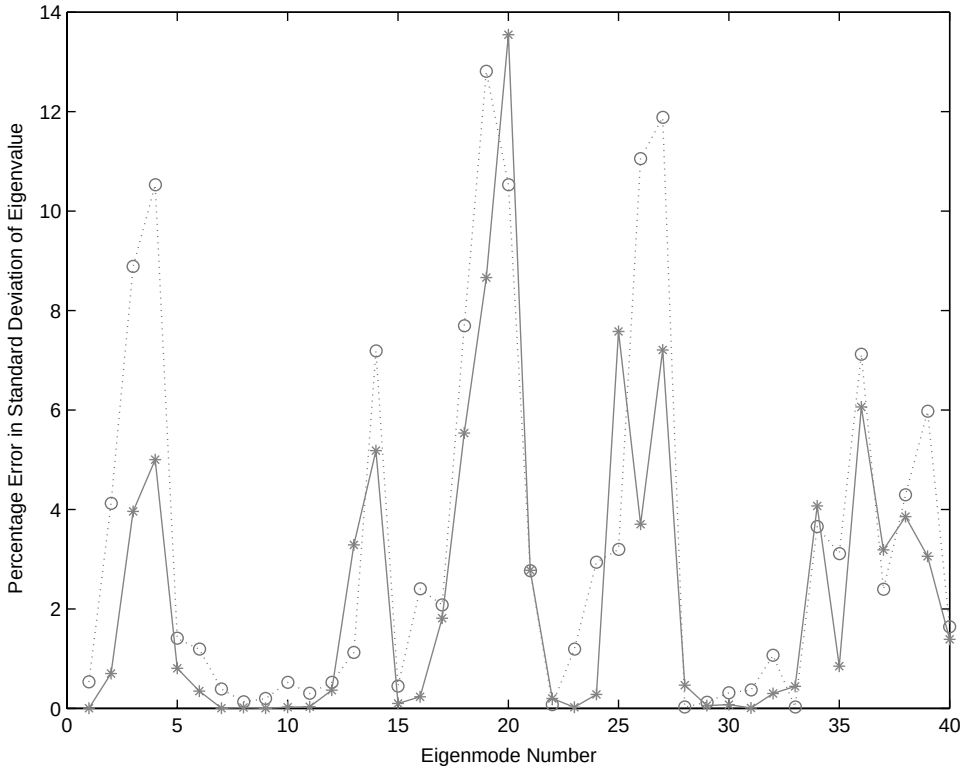

Fig. 4. Comparison of errors in standard deviation of the natural frequencies for case 1, $\sigma_\theta = 0.05$. -o-, PM1; -∗-, RBA.

range of 1 to p. It can readily be seen from Table 1 that when the system matrices are symmetric, the second and third order tensors in Table 1 will also be symmetric.

Using this formulation, the eigenvalues of the reduced order random eigenvalue problem can be computed by solving for the roots of the quadratic

$$(a_{11}a_{22} - b_{12}^2)\lambda^2 + (2a_{12}b_{12} - a_{11}b_{22} - a_{22}b_{11})\lambda = a_{11}a_{22} - a_{12}^2, \tag{19}$$

where a_{ij} and b_{ij} denote the elements of the reduced random matrices $\mathbf{A}_R(\boldsymbol{\theta})$ and $\mathbf{B}_R(\boldsymbol{\theta})$, respectively. For the sake of notational convenience, the dependence of these elements on the random variables is not explicitly shown. Note that the quadratic in Eq. (19) will give two possible values for the approximate eigenvalue. Clearly, for the fundamental eigenmode, the root with the minimum value gives the best approximation. For the higher modes the best approximation is chosen by selecting the root which is closest to the higher order eigenvalue approximation proposed in Ref. [25], which can be written using tensor notation as

$$\hat{\lambda} = \lambda_o + \frac{[\theta_i a_i + \theta_i\theta_j E_{ij} - \lambda^o(\theta_i b_i + \theta_i\theta_j F_{ij})]}{[1 + \theta_i d_i + \theta_i b_i + \theta_i\theta_j F_{ij}]}. \tag{20}$$

The selection of the appropriate root based on this criteria can be carried out by transforming Eq. (19) using the substitution $\lambda = \gamma + \hat{\lambda}$, which gives the modified quadratic

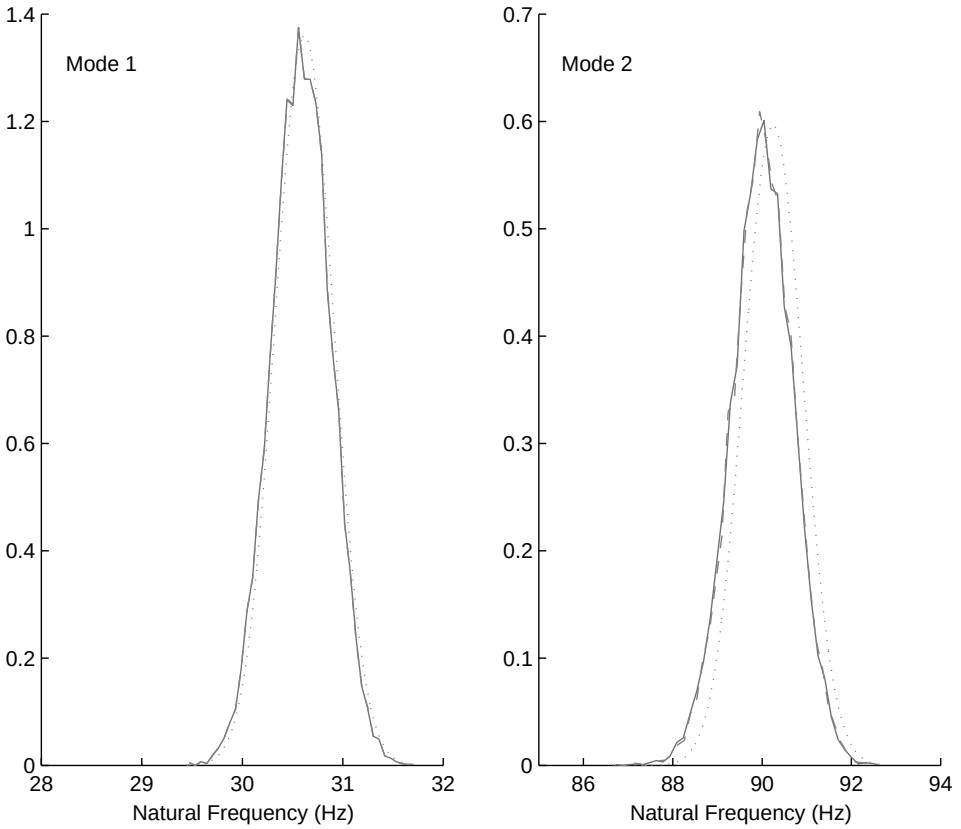

Fig. 5. Comparison of pdf of the first two natural frequencies for case 1, $\sigma_\theta = 0.05$. —-, MCS; - - -, RBA; ⋯, PM1.

equation

$$(a_{11}a_{22} - b_{12}^2)(\gamma + \hat{\lambda})^2 + (2a_{12}b_{12} - a_{11}b_{22} - a_{22}b_{11})(\gamma + \hat{\lambda}) = a_{11}a_{22} - a_{12}^2. \tag{21}$$

The best approximation is hence that root of Eq. (21) which has smallest absolute value, i.e., $\min(\gamma)$. The random eigenvalue can hence be evaluated as $\lambda(\boldsymbol{\theta}) = \min(\gamma) + \hat{\lambda}$. Using this approximation for the eigenvalue, the random eigenvector is approximated such that it satisfies the normalization condition with respect to $[\mathbf{B}_o + \sum_{i=1}^{p} \theta_i \mathbf{B}_i]$ with probability one. After some further algebra, an approximation for the normalized random eigenvector can be written as

$$\hat{\mathbf{x}}(\boldsymbol{\theta}) = \frac{1}{\sqrt{\beta}}\left[\mathbf{x}^o + \left(\frac{a_{11} - \lambda(\boldsymbol{\theta})b_{11}}{a_{12} - \lambda(\boldsymbol{\theta})b_{12}}\right)\sum_{i=1}^{p} \theta_i \frac{\partial \mathbf{x}}{\partial \theta_i}\right], \tag{22}$$

where

$$\beta = b_{11} + \left(\frac{a_{11} - \lambda(\boldsymbol{\theta})b_{11}}{a_{12} - \lambda(\boldsymbol{\theta})b_{12}}\right)b_{12} + \left(\frac{a_{11} - \lambda(\boldsymbol{\theta})b_{11}}{a_{12} - \lambda(\boldsymbol{\theta})b_{12}}\right)^2 b_{22}. \tag{23}$$

Conceptually, the statistics of the eigenvalues and eigenvectors can be computed using Eqs. (20)–(23). However, since the resulting expressions for the eigenvalues and eigenvectors are

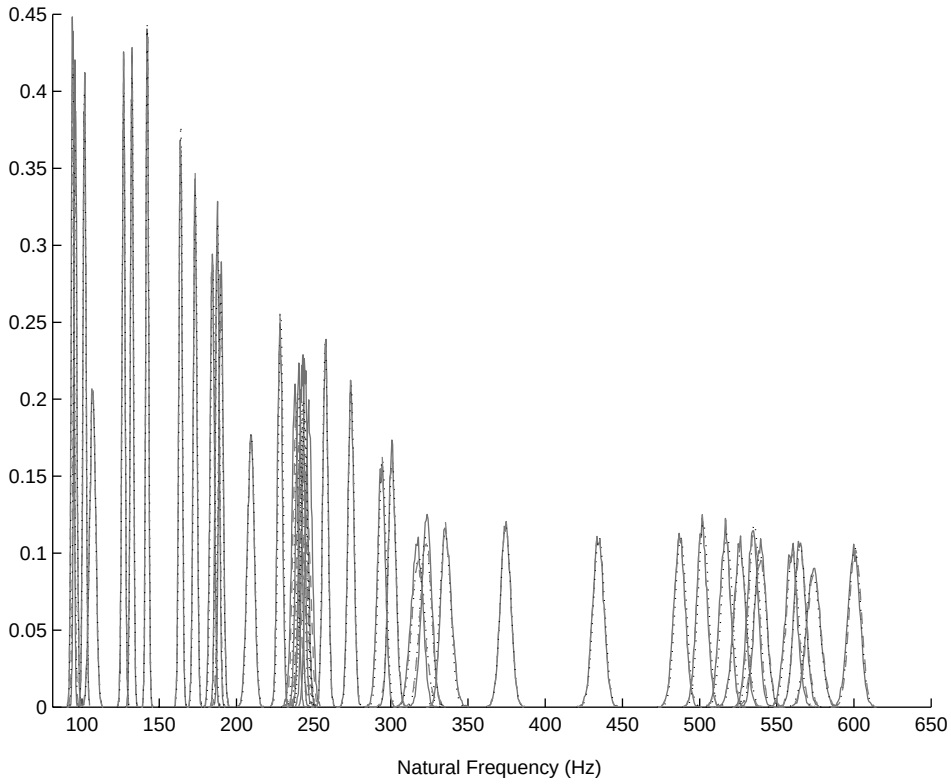

Fig. 6. Comparison of pdf's of natural frequencies corresponding to modes 3–40 for case 1, $\sigma_\theta = 0.05$. —-, MCS; - - -, RBA; ⋯, PM1.

highly non-linear functions of the random variables, analytical solutions for the statistical moments are not readily possible. Fortunately, the solution of Eqs. (20)–(23) requires only $\mathcal{O}(p^3)$ operations for a given realization of $\boldsymbol{\theta}$. Hence, a complete probabilistic description of the eigenvalues and eigenvectors is within reach using simulation techniques. The formulation presented in this section is henceforth referred to as RBA.

4.1. Remarks

Note that two different lines of approach can be employed to set up the reduced order random eigenvalue problem. For example, one could use a global set of stochastic basis vectors to simultaneously approximate all the desired eigenvalues and eigenvectors. This approach would entail the use of the appropriate inner product defined in the Hilbert space of random variables. Clearly, this approach would be desirable for systems with high modal density. However, the formulation presented here uses independent sets of two basis vectors to approximate each eigenmode of interest. The implicit assumption made here is that the effects of mode-switching will not lead to significant approximation errors. This suggests that the present method may run into difficulties when the statistical overlap factor (sof) is high. The sof defined earlier by Manohar

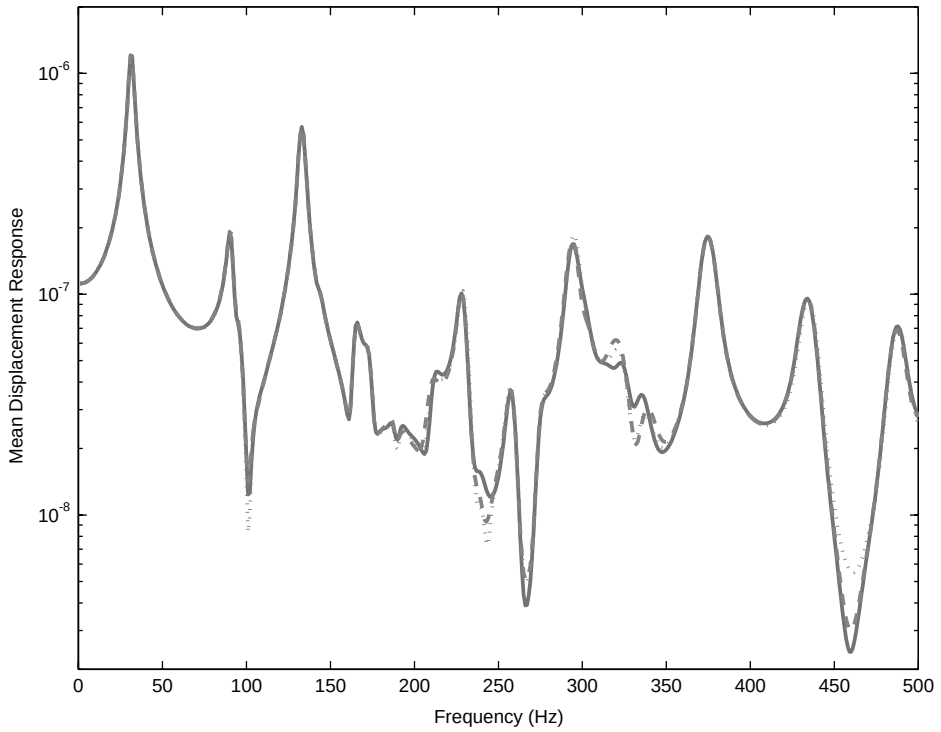

Fig. 7. Comparison of mean displacement response for case 1, $\sigma_\theta = 0.05$. —-, MCS; - - -, RBA; ⋯, PM1.

and Keane [26] is given by.

$$sof_i = 2\frac{\sigma_i}{\mu_i},$$

where σ_i is the standard deviation of the ith natural frequency, and μ_i denotes the mean spacing between the ith and $(i+1)$th natural frequency.

However, as shown later via numerical studies, if the approximate eigenvalues are appropriately reordered, reasonable approximations for the eigenmodes with high sof values can be computed.

4.2. Simplification of the formulation

This section examines some approximations to simplify the reduced basis method and to improve the computational efficiency. As mentioned earlier, $\mathcal{O}(p^3)$ operations are required to approximate the eigenvalue and eigenvector of each mode for a given realization of the random variables. Further, it can also be observed from Table 1 that $\mathcal{O}(p^3)$ scalars are required to be stored in memory for each eigenmode. This may become rather high for systems with large number of uncertain parameters. In order to reduce the computational cost and memory requirements, the third order tensors which appear in the expressions for $\mathbf{A}_R(\boldsymbol{\theta})$ and $\mathbf{B}_R(\boldsymbol{\theta})$ can be simplified. Consider, for example, a typical term involving third order terms

$$a_{22} = \theta_i\theta_j\mathbf{G}_{ij} + \theta_i\theta_j\theta_k\mathcal{Q}_{ijk}. \tag{24}$$

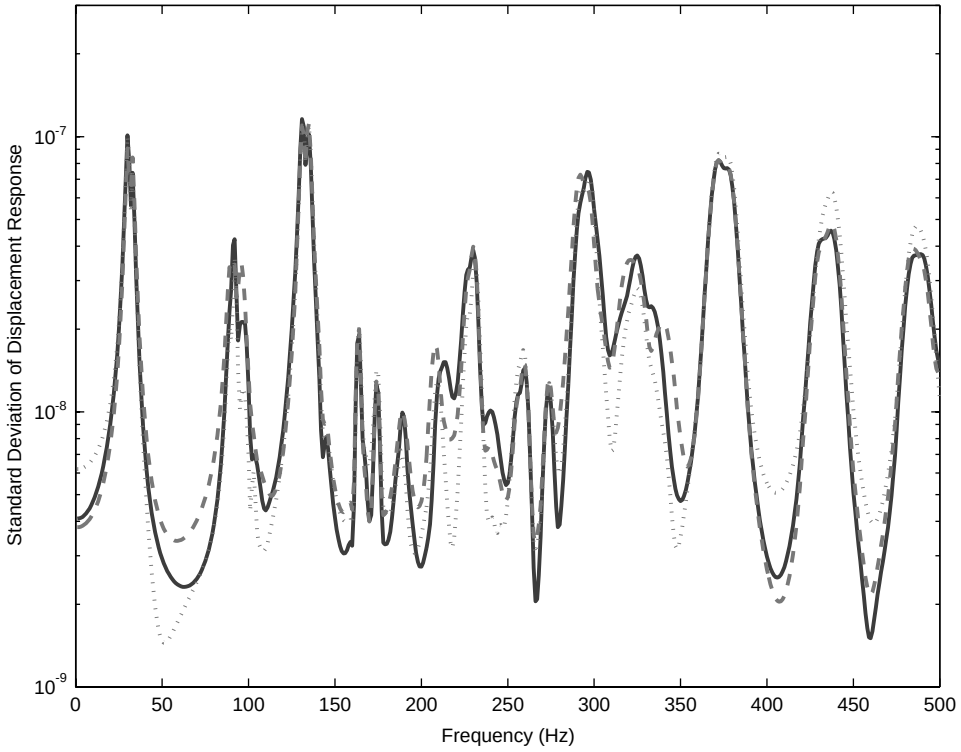

Fig. 8. Comparison of standard deviation of displacement response for case 1, $\sigma_\theta = 0.05$. —-, MCS; - - -, RBA; ⋯, PM1.

This term can be simplified by replacing the third order term with its ensemble average, i.e.,

$$\bar{a}_{22} = \theta_i\theta_j\mathbf{G}_{ij} + \langle \theta_i\theta_j\theta_k \rangle \mathcal{Q}_{ijk}. \tag{25}$$

Similarly, the third order term appearing in b_{22} can also be replaced by its ensemble average. The expectation operation in Eq. (25) can be readily computed using the probability density function (pdf) of the vector $\boldsymbol{\theta}$. This simplification allows for the solution of the reduced order random eigenvalue problem using only $\mathcal{O}(p^2)$ operations for a given realization of the random system parameters.

4.3. A note on computational aspects

The steps involved in the present formulation are summarized below:

Step 1: The first step involves representing the random eigenvalue in the form of Eq. (6). This can readily be done either by discretizing the underlying random fields of the governing SDEP, or by computing the sensitivities of the system matrices with respect to the random physical parameters.

Step 2: The deterministic eigenvalue problem in Eq. (7) is solved for the eigenmodes of interest, and the eigenvector derivatives are computed with respect to the random variables.

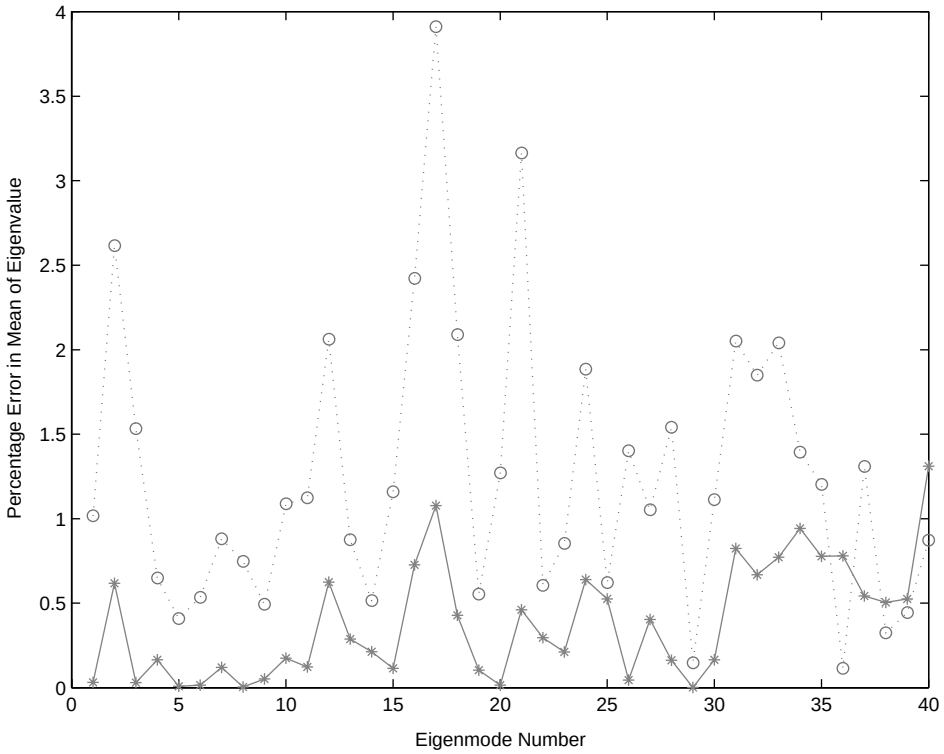

Fig. 9. Comparison of errors in mean of natural frequencies for case 2, $\sigma_\theta = 0.15$. -o-, PM1; -∗-, RBA.

Step 3: The problem specific deterministic tensors given in Table 1 are computed for each eigenmode of interest. The computational complexity of this step is $\mathcal{O}(n^2)$, since only matrix vector multiplications are involved.

Step 4: The constants computed in Step 3 are then used to expedite the statistical analysis of the eigenvalues and eigenvectors via MCS using Eq. (20)–(23). Note that the eigenvalues have to be appropriately reordered at this stage to ensure that the eigenvalue statistics are of reasonable accuracy.

It is of interest to note that the statistics of the eigenvalues and eigenvectors of each eigenmode can be computed independently of each other. This enables the possibility of leveraging parallel computing systems for solving large-scale random eigenvalue problems. Once the eigenvalues and eigenvectors are approximated for a given realization of $\boldsymbol{\theta}$, the forced response can be readily computed. This enables the efficient computation of the forced response statistics.

5. Demonstration examples, results, and discussion

Numerical studies are presented for free and forced vibration analysis of the network of stochastic Euler–Bernoulli beams with random Young's modulus shown in Fig. 1. The structure is modelled using 3 elements for each beam member, which leads to a finite element model with a

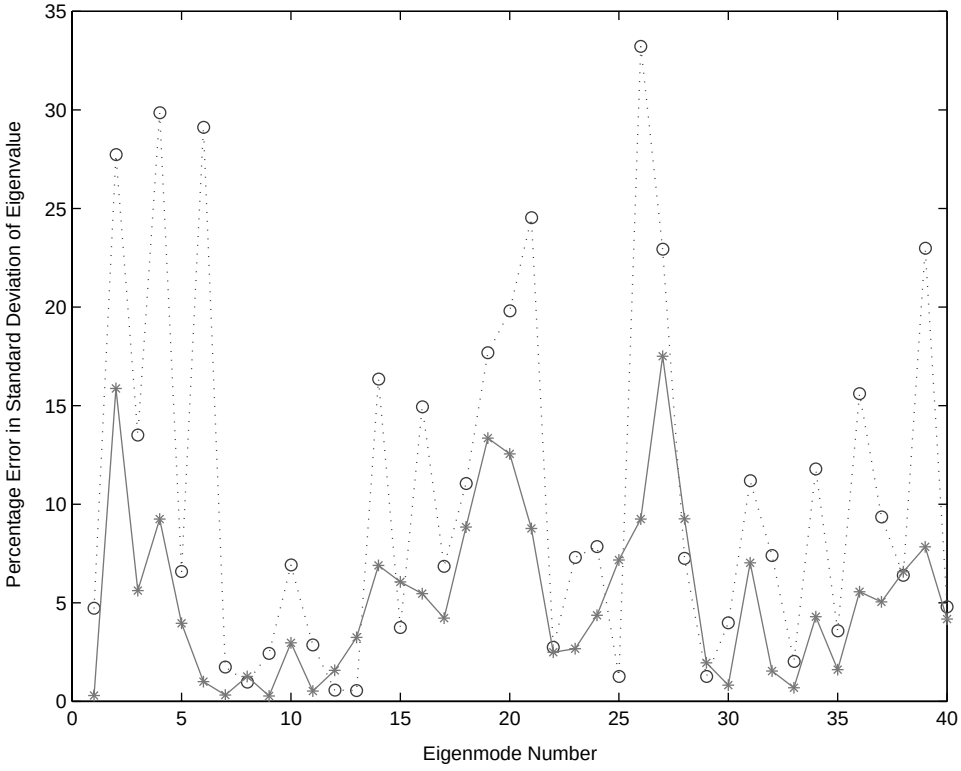

Fig. 10. Comparison of errors in standard deviation of the natural frequencies for case 2, $\sigma_\theta = 0.15$. -o-, PM1; -∗-, RBA.

total of 150 degrees of freedom. The axial and flexural rigidity of each structural member are modelled as $E^0A(1+\theta_i)$ and $E^0I(1+\theta_i)$, $i = 1, 2, \ldots, 20$. θ_i are considered as uncorrelated zero-mean Gaussian random variables with standard deviation σ_θ, while $E^0A = 6.987 \times 10^6$ N, $E^0I = 1.286 \times 10^3$ Nm^2, and the mass density $\rho = 2.74$ kg/m. This leads to a total of 20 random system parameters for this problem.

Two cases are considered to compare the accuracy of the methods when σ_θ is increased. The value of σ_θ is set at 0.05, and 0.15 for cases 1 and 2, respectively. Numerical studies were conducted to compute the statistics of the first 40 eigenmodes, and the transverse component of the displacement response at node 9 in the frequency range of 0–500 Hz, when the structure is subjected to transverse harmonic excitation at node 1.

MCS using exact eigensolution with a sample size of 10,000 is used to generate benchmark results against which the reduced basis formulation and the first order perturbation method are compared. The benchmark results are referred to as exact results throughout the discussion. For the reduced basis approximation (RBA) method, the response statistics were computed using the same sample size. Note that for the first order perturbation method (PM1), the eigenvalue statistics can be computed analytically, since a linear approximation is involved. Similarly, it is also possible to compute the forced response statistics analytically using a first order sensitivity analysis. However, a simulation scheme with a sample size of 10,000 was used

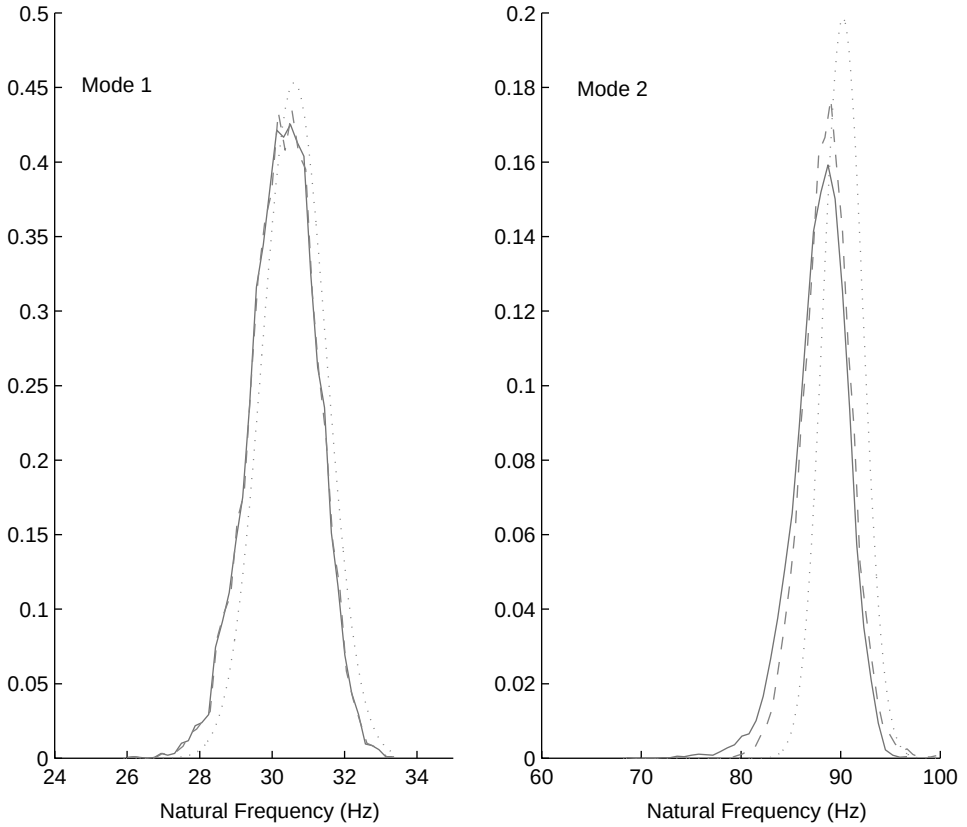

Fig. 11. Comparison of pdf of the first two natural frequencies for case 2, $\sigma_\theta = 0.15$. —-, MCS; - - -, RBA; (···), PM1.

here to compute the forced response statistics, using the eigenvalues and eigenvectors approximated using PM1.

Fig. 2 shows the trends of the sof of all the natural frequencies when σ_θ is kept at 0.05, 0.10, and 0.15. It can be observed that the sof of all the modes increase appreciably when σ_θ is changed from 0.05 to 0.15. This suggests that the example problem considered is fairly challenging since the modes are highly likely to switch in the presence of parametric uncertainties.

A comparison of the percentage errors in the mean and standard deviation of the natural frequencies for case 1 using RBA and PM1 are shown in Figs. 3 and 4. It can be seen that with the standard deviation of 0.05, reasonably accurate results can be obtained for the first two statistical moments of most of the eigenvalues using the approximate methods. The accuracy of RBA is seen to be better than PM1 for most of the eigenmodes of interest. A comparison of Fig. 2 with Figs. 3 and 4 shows that the errors in the statistical moments tend to be higher for the natural frequencies with high sof.

The pdf's of the first two natural frequencies computed using all the methods are shown in Fig. 5. The pdfs of the natural frequencies corresponding to modes 3–40 are shown in Fig. 6. The pdf plots clearly show the degree of overlap due to parameter uncertainties. It also appears from the figures that the pdf's can be well approximated by a Gaussian distribution. The mean and

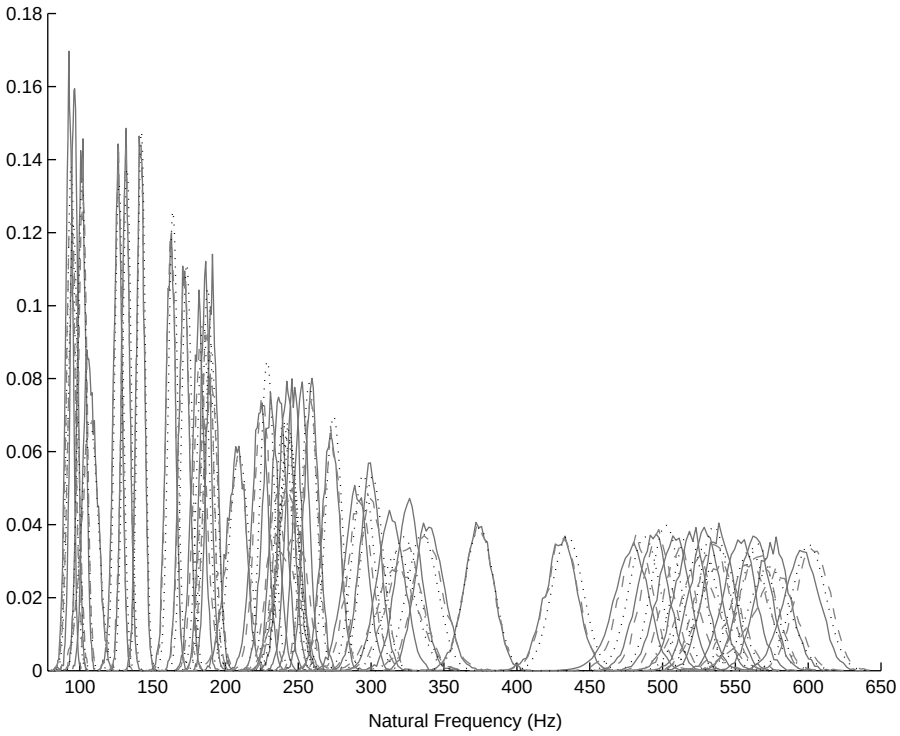

Fig. 12. Comparison of pdf's of natural frequencies corresponding to modes 3–40 for case 2, $\sigma_\theta = 0.15$. —-, MCS; - - -, RBA; ⋯, PM1.

standard deviation of the frequency response computed using RBA and PM1 are compared with the exact results in Figs. 7 and 8. It can be observed that RBA shows good correlation with the exact results for both statistical moments of the frequency response. In comparison, the errors in PM1 are higher at some frequency points. These trends indicate that the eigenvector statistics computed using RBA are more accurate than PM1.

A comparison of the percentage errors in the mean and standard deviation of the natural frequencies computed using RBA and PM1 for case 2 are shown in Figs. 9 and 10. As observed earlier for case 1, RBA gives better results as compared to PM1 for the statistical moments of most of the eigenvalues. However, for some eigenmodes, the mean and standard deviation predicted by PM1 can be seen to be marginally more accurate than RBA.

The pdf's of the first two natural frequencies computed using various methods are compared in Fig. 11. It can be seen that RBA shows good agreement with MCS for the fundamental natural frequency. The pdf's of the natural frequencies corresponding to modes 3–40 are shown in Fig. 12. As compared to case 1, it can be observed that the pdfs corresponding to almost all the natural frequencies show a greater degree of overlap.

The mean and standard deviation of the frequency response computed using RBA and PM1 are compared with the exact results in Figs. 13 and 14. It can be seen that RBA shows better agreement with the exact results for both the mean and the standard deviation of the response. This suggests that RBA gives better approximations for the eigenvector statistics as compared to

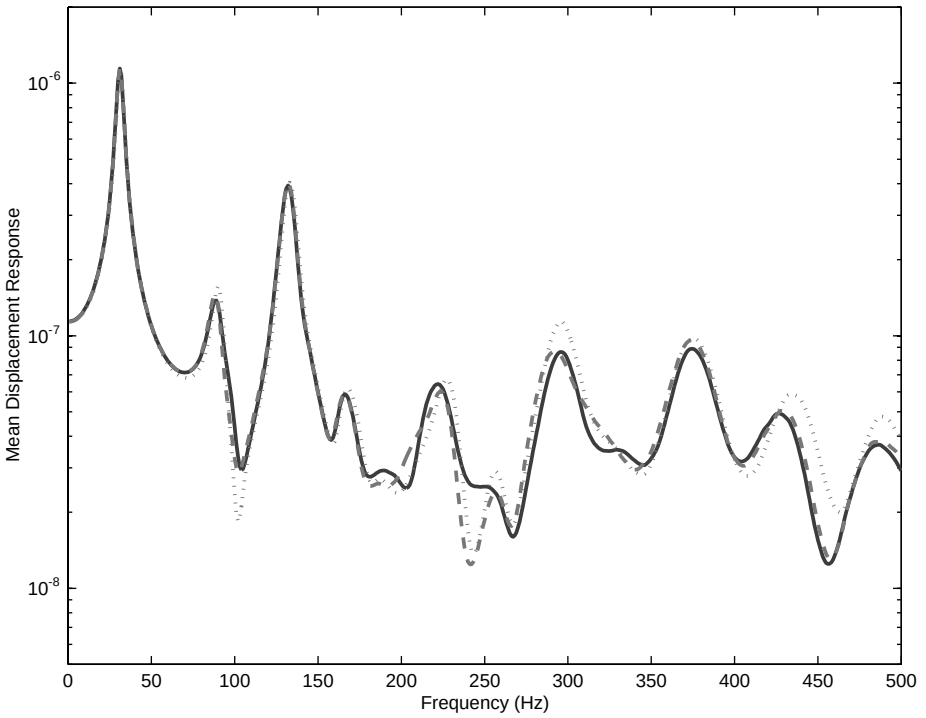

Fig. 13. Comparison of mean displacement response for case 2, $\sigma_\theta = 0.15$. —-, MCS; - - -, RBA; ···, PM1.

PM1. It is of interest to note that, in spite of the high magnitude of errors in the eigenvalue statistics, the first two statistical moments of the frequency response are of reasonable accuracy.

Finally, it is worth noting that computation of the first 40 modes of this structural system using the Lanczos method requires around 1.7 s on one node of an SGI Origin2000 with R10,000 processors. Hence, the MCS procedure using a sample size of 10,000 involved nearly 5 h of processor time. In contrast, the reduced basis formulation required only around 2.3 min, with the first order perturbation method taking around 1.7 min. Note that the routines implementing the approximate methods have not been fully optimized. It is expected that the difference between the computational cost of the methods will become even more significant with increase in the problem size, i.e., both RBA and PM1 will require only a very small fraction of the computation cost required for MCS using exact eigensolution.

6. Concluding remarks

A stochastic reduced basis formulation is presented for efficiently solving large-scale algebraic random eigenvalue problems. In this formulation, the original eigenvalue problem is reduced into a sequence of (2×2) reduced order random eigenvalue problems for each mode of interest. The terms of the reduced order eigenvalue problem can be efficiently computed by solving a

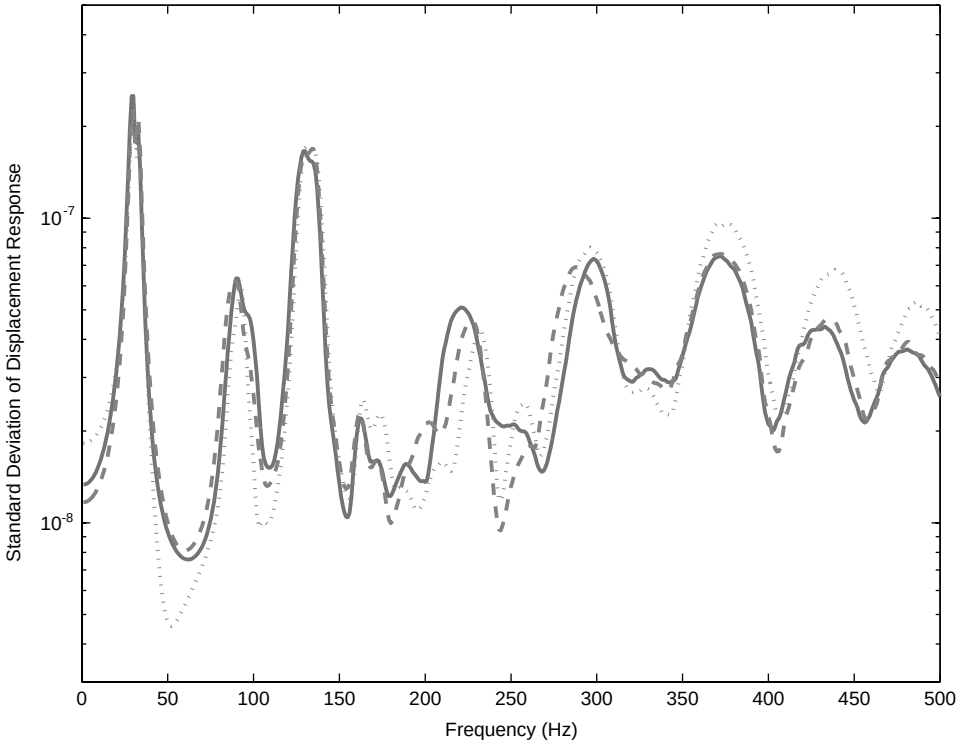

Fig. 14. Comparison of standard deviation of displacement response for case 2, $\sigma_\theta = 0.15$. —-, MCS; - - -, RBA; ⋯, PM1.

deterministic eigenvalue problem and computing its sensitivities. Further, the present formulation allows explicit expressions for the random eigenvalues and eigenvectors to be derived in terms of the random system properties. This enables the statistics of the random eigenparameters and the forced response to be efficiently computed.

Numerical studies have been presented for free and forced vibration analysis of a linear stochastic structural system to demonstrate that improvements over the first order perturbation method can be achieved, particularly for moderate stochastic variations in the system properties. In particular, reasonably accurate results can be obtained for the eigenvalue statistics when the statistical overlap factor is low. As compared to the first order perturbation method, the reduced basis method gives better approximations for the first two statistical moments of the frequency response. In fact, the statistics of the frequency response are observed to be reasonably accurate even when the errors in the statistical moments of the eigenvalues are rather high. It is also demonstrated that the improvement in accuracy over the first order perturbation method is achieved with only a small increment in the computational effort.

Preliminary investigations suggest that formulations which use a global set of stochastic basis vectors to simultaneously approximate all the desired eigenvalues and eigenvectors [27] may lead to more accurate results. Such general stochastic subspace projection schemes are also expected to allow for the possibility of analytically approximating the forced response statistics without using simulation techniques. It is also of interest to note that the present method can be extended to

linear algebraic random eigenvalue problems with general non-Hermitian matrices, and quadratic random eigenvalue problems.

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References

[1] R. Ghanem, P. Spanos, Stochastic Finite Elements: A Spectral Approach, Springer, Berlin, 1991.
[2] C.S. Manohar, R.A. Ibrahim, Progress in structural dynamics with stochastic parameter variations, Applied Mechanics Reviews 52 (5) (1999) 177–197.
[3] R.S. Langley, The dynamic analysis of uncertain structures. Proceeding of the Seventh International Conference on Recent Advances in Structural Dynamics, Vol. 1, 2000, pp. 1–20.
[4] B.F. Spencer Jr, M.K. Sain, J.C. Kantor, C. Montemagno, Probabilistic stability measures for controlled structures subject to real parameter uncertainties, Smart Materials and Structures 1 (1999) 294–305.
[5] L. Papadopoulos, E. Garcia, Structural damage identification: a probabilistic approach, American Institute of Aeronautics and Astronautics Journal 36 (11) (1998) 2137–2145.
[6] C. Pierre, M.P. Castanier, S.B. Choi, On developing new statistical energy methods for the analysis of vibration transmission in complex vehicle structures, Mechanics of Structures and Machines 25 (1) (1997) 87–101.
[7] A. Edelman, Eigenvalues and Condition Numbers of Random Matrices, Ph.D. Dissertation, Massachusetts Institute of Technology.
[8] R.S. Langley, A non-Poisson model for the vibration analysis of uncertain dynamic systems, Proceedings of the Royal Society of London, Series A Mathematical, Physical, and Engineering Sciences 455 (1989) 3325–3349.
[9] J.D. Collins, W.T. Thompson, The eigenvalue problem for structures with statistical properties, American Institute of Aeronautics and Astronautics Journal 7 (4) (1969) 642–648.
[10] M. Kleiber, T.D. Hien, The Stochastic Finite Element Method: Basic Perturbation Technique and Computer Implementation, Wiley, Chichester, 1992.
[11] C. Lee, R. Singh, Analysis of discrete vibratory systems with parameter uncertainties, Part I: eigensolution, Journal of Sound and Vibration 174 (3) (1999) 379–394.
[12] M. Grigoriu, A solution of the random eigenvalue problem by crossing theory, Journal of Sound and Vibration 158 (1) (1992) 69–80.
[13] J.R. Red-Horse, R. Ghanem, Polynomial chaos representation of the random eigenvalue problem, Proceedings of the 40th American Institute of Aeronautics and Astronautics/ASME/ASCE/AHS/ ASC Structures, Structural Dynamics, and Materials Conference, St. Louis, MO, 1999.
[14] G. Ottarson, Dynamic Modeling and Vibration Analysis of Mistuned Bladed Disks, Ph.D. Dissertation, University of Michigan, 1999.
[15] P.B. Nair, A.J. Keane, New developments in computational stochastic mechanics, part I: theory. Proceedings of the 41st American Institute of Aeronautics and Astronautics/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Atlanta, GA, American Institute of Aeronautics and Astronautics Paper 2000-1827, 2000.
[16] P.B. Nair, A.J. Keane, New developments in computational stochastic mechanics, part II: applications. Proceedings of the 41st American Institute of Aeronautics and Astronautics/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Atlanta, GA, American Institute of Aeronautics and Astronautics Paper 2000-1441, 2000.

[17] P.B. Nair, A.J. Keane, Stochastic reduced basis methods, American Institute of Aeronautics and Astronautics Journal 40 (8) (2002) 1653–1664.
[18] P.B. Nair, A.J. Keane, R.S. Langley, Improved first order approximation of eigenvalues and eigenvectors, American Institute of Aeronautics and Astronautics Journal 36 (9) (1721–1727).
[19] D.V. Murthy, R.T. Haftka, Approximations to eigenvalues of modified general matrices, Computers and Structures 29 (5) (1988) 903–917.
[20] C.-C. Li, A. Der Kiureghian, Optimal discretization of random fields, ASCE Journal of Engineering Mechanics 119 (6) (1993) 1136–1154.
[21] H. Mei, O.P. Agarwal, S.S. Pai, Wavelet-based model for stochastic analysis of beam structures, American Institute of Aeronautics and Astronautics Journal 36 (3) (1998) 465–470.
[22] O. Zhang, A. Zerva, Accelerated iterative procedure for calculating eigenvector derivatives, American Institute of Aeronautics and Astronautics Journal 35 (2) (1997) 340–348.
[23] D.W. Zhang, F.S. Wei, Structural eigenderivative analysis using practical and simplified dynamic flexibility method, American Institute of Aeronautics and Astronautics Journal 37 (7) (1999) 865–873.
[24] M. Akgün, New family of modal methods for computing eigenvector derivatives, American Institute of Aeronautics and Astronautics Journal 32 (2) (1994) 379–386.
[25] M.S. Eldred, P.B. Lerner, W.J. Anderson, Higher order eigenpair perturbations, American Institute of Aeronautics and Astronautics Journal 30 (7) (1992) 1870–1876.
[26] C.S. Manohar, A.J. Keane, Statistics of energy flows in spring-coupled one-dimensional sub-systems, Philosophical Transactions of the Royal Society of London 346 (A) (1994) 525–542.
[27] P.B. Nair, On the theoretical foundations of stochastic reduced basis methods. Proceedings of the 42nd American Institute of Aeronautics and Astronautics/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference, Seattle, WA, American Institute of Aeronautics and Astronautics Paper No. 2001-1677, 2001.